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Dynamics of decoherence without dissipation in a squeezed thermal bath

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Abstract

We study a generic open quantum system where the coupling between the system and its environment is of an energy-preserving quantum nondemolition (QND) type. We obtain the general master equation for the evolution of such a system under the influence of a squeezed thermal bath of harmonic oscillators. From the master equation it can be seen explicitly that the process involves decoherence or dephasing without any dissipation of energy. We work out the decoherence-causing term in the high- and zero-temperature limits and check that they match with known results for the case of a thermal bath. The decay of the coherence is quantified as well by the dynamics of the linear entropy of the system under various environmental conditions. We make a comparison of the quantum statistical properties between QND and dissipative types of evolution using a two-level atomic system and a harmonic oscillator.

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1. Introduction

The concept of 'open' quantum systems is a ubiquitous one in that all real systems of interest are open systems, each surrounded by its environment, which affects its dynamics. Caldeira and Leggett [1] used the influence functional approach developed by Feynman and Vernon [2] to discuss quantum dissipation via the paradigm of quantum Brownian motion (QBM) of a simple harmonic oscillator in a harmonic oscillator environment. The influence of the environment on the reduced dynamics of the system was quantified by the influence functional. Dissipation of the system originates from the transfer of energy from the system of interest to the 'large' environment. The energy, once transferred, is not given back to the system within any time of physical relevance. In the original model of the QBM, the system and its environment were taken to be initially uncorrelated. The treatment was extended to the physically reasonable initial condition of a mixed state of the system and its environment by

Hakim and Ambegaokar [3], Smith and Caldeira [4], Grabert *et al* [5] and Banerjee and Ghosh [6] among others. Haake and Reibold [7] and Hu *et al* [8] obtained an exact master equation for the quantum Brownian particle for a general spectral density of the environment.

The spectacular progress in the manipulation of quantum states of matter and applications in quantum information processing have resulted in a renewed demand for understanding and control of the environmental impact in such open quantum systems. For such systems, there exists an important class of energy-preserving measurements in which dephasing occurs without damping of the system. This may be achieved with a particular type of coupling between the system and its environment, namely, when the Hamiltonian H_S of the system commutes with the Hamiltonian H_{SR} describing the system–reservoir interaction, i.e., H_{SR} is a constant of motion generated by H_S [9–11]. This condition describes a particular type of quantum nondemolition (QND) measurement scheme.

In general, a class of observables that may be measured repeatedly with an arbitrary precision, with the influence of the measurement apparatus on the system being confined strictly to the conjugate observables, is called QND or back-action evasive observables [12–16]. Such a measurement scheme was originally introduced in the context of the detection of gravitational waves [17, 18]. The dynamics of decoherence in continuous QND measurements applied to a system of two-level atom interacting with a stationary quantized electromagnetic field through a dispersive coupling has been studied by Onofrio and Viola [19]. In addition to its relevance in ultrasensitive measurements, a QND scheme provides a way to prepare quantum mechanical states which may otherwise be difficult to create—such as Fock states with a specific number of particles. It has been shown that the accuracy of atomic interferometry can be improved by using QND measurements of the atomic populations at the inputs to the interferometer [20]. QND systems have also been proposed for engineering quantum dynamical evolution of a system with the help of a quantum meter [21]. We have recently studied such QND open system Hamiltonians for two different models of the environment describable as baths of either oscillators or spins, and found an interesting connection between the energy-preserving QND Hamiltonians and the phase space area-preserving canonical transformations [22].

In this paper, we wish to study the dynamics of decoherence in a generic open quantum system where the coupling between the system and its environment is of the energy-preserving QND type. The bath is taken to be initially in a squeezed thermal state, from which the common thermal bath results may be easily extracted by setting the squeezing parameters to zero. When the quantum fluctuations of the heat bath are squeezed, it has been shown by Kennedy and Walls [23] that the macroscopic superposition of states of light is preserved in the presence of dissipation. These authors have shown that the squeezed bath is more efficient than the thermal bath for optical quadrature-phase quantum measurements, and may also be used to prepare the states with low quantum noise in one quadrature phase, at least in the high-frequency regime. The advantage of using a squeezed thermal bath over an ordinary phase-insensitive thermal bath is that the decay rate of quantum coherences can be suppressed in a squeezed bath leading to preservation of nonclassical effects [24]. Such a bath has also been shown to modify the evolution of the geometric phase of two-level atomic systems [25]. In our present problem, we wish to systematically probe the effect of phase-sensitivity of the bath, and quantify the pattern of progressive decay of coherence of the system, both at high temperatures as well as arbitrary low temperatures, when a quantum nondemolition coupling is adopted. We wish to hence compare and contrast the quantum statistical mechanical features (namely, the nature of the noise channels) of the QND type of evolution with that of the dissipative evolution [8, 26–29] of the general Lindblad form for a two-level system or the specific QBM form for a harmonic oscillator.

The plan of the paper is as follows. In section 2, we obtain the master equation for a generic system interacting with its environment by a QND type of coupling. For simplicity, we take the system and its environment to be initially separable. In section 2.1, the master equation is obtained for the case of a bosonic bath of harmonic oscillators. For the sake of completeness, we briefly compare the results for the oscillator bath with that for a bath of two-level systems in section 2.2. In section 3, we analyze the dynamics of decoherence, first by looking at the term causing decoherence in the system master equation for the bosonic bath of harmonic oscillators obtained in section 2.1, and explicitly solve it for the high-temperature and the zero-temperature cases. We then set up a quantitative 'measure of coherence' related to the linear entropy $S(t)$, the dynamics of which is also studied for the zero- as well as the high-temperature cases for different degrees of squeezing of the bath. In section 4, the quantum statistical mechanical properties underlying the QND and dissipative processes are studied on a general footing for a two-level atomic system (section 4.1) and a harmonic oscillator system (section 4.2). For the two-level system, the dissipative process is taken to be that generated by a standard Lindblad equation while for the harmonic oscillator system, the model studied is that of the QBM. In section 5, we present our conclusions.

2. Master equation

Here we present the master equation for a system interacting with its environment by a coupling of the energy-preserving QND type where the environment is a bosonic bath of harmonic oscillators initially in a squeezed thermal state, decoupled from the system. We also take up the case where the environment is composed of a bath of two-level systems, and compare the two cases.

2.1. Bath of harmonic oscillators

We consider the Hamiltonian

$$\begin{aligned} H &= H_S + H_R + H_{SR} \\ &= H_S + \sum_k \hbar\omega_k b_k^\dagger b_k + H_S \sum_k g_k \left(b_k + b_k^\dagger\right) + H_S^2 \sum_k \frac{g_k^2}{\hbar\omega_k}. \end{aligned} \tag{1}$$

Here, H_S, H_R and H_{SR} stand for the Hamiltonians of the system, reservoir and system–reservoir interaction, respectively. H_S is a generic system Hamiltonian which can be specified depending on the physical situation. $b_k^\dagger$ and b_k denote the creation and annihilation operators, respectively, for the reservoir oscillator of frequency ω_k, and g_k stands for the coupling constant (assumed real) for the interaction of the oscillator field with the system. The last term on the right-hand side of equation (1) is a renormalization inducing 'counter term'. Since $[H_S, H_{SR}] = 0$, the Hamiltonian (1) is of QND type. The system-plus-reservoir composite is closed and hence obeys a unitary evolution given by

$$\rho(t) = \mathrm{e}^{-\mathrm{i}Ht/\hbar}\rho(0)\,\mathrm{e}^{\mathrm{i}Ht/\hbar}, \tag{2}$$

where

$$\rho(0) = \rho^s(0)\rho_R(0), \tag{3}$$

i.e., we assume separable initial conditions. In order to obtain the reduced dynamics of the system alone, we trace over the reservoir variables. The matrix elements of the reduced density matrix in the system eigenbasis are

$$\rho^s_{nm}(t) = \mathrm{e}^{-\mathrm{i}(E_n-E_m)t/\hbar}\,\mathrm{e}^{-\mathrm{i}(E_n^2-E_m^2)/\hbar\sum_k(g_k^2 t/\hbar\omega_k)}\,\mathrm{Tr_R}[\mathrm{e}^{-\mathrm{i}H_n t/\hbar}\rho_R(0)\,\mathrm{e}^{\mathrm{i}H_m t/\hbar}]\rho^s_{nm}(0), \tag{4}$$

where E_n's are the eigenvalues of the system Hamiltonian, $\rho_R(0)$ is the initial density matrix of the reservoir which we take to be a squeezed thermal bath given by

$$\rho_R(0) = S(r, \Phi)\rho_{\text{th}} S^{\dagger}(r, \Phi), \tag{5}$$

where

$$\rho_{\text{th}} = \prod_k [1 - \mathrm{e}^{-\beta\hbar\omega_k}]\, \mathrm{e}^{-\beta\hbar\omega_k b_k^{\dagger} b_k} \tag{6}$$

is the density matrix of the thermal bath at temperature T, with $\beta \equiv 1/(k_B T)$, k_B being the Boltzmann constant, and

$$S(r_k, \Phi_k) = \exp\left[r_k\left(\frac{b_k^2}{2}\,\mathrm{e}^{-2\mathrm{i}\Phi_k} - \frac{b_k^{\dagger 2}}{2}\,\mathrm{e}^{2\mathrm{i}\Phi_k}\right)\right] \tag{7}$$

is the squeezing operator with r_k, Φ_k being the squeezing parameters [30]. In equation (4),

$$H_n = \sum_k \left[\hbar\omega_k b_k^{\dagger} b_k + E_n g_k\left(b_k + b_k^{\dagger}\right)\right]. \tag{8}$$

Following the steps of the derivation as shown in the appendix, the reduced density matrix (4) of the system is obtained as

$$\begin{aligned} \rho^s_{nm}(t) = {} & \mathrm{e}^{-\mathrm{i}(E_n - E_m)t/\hbar}\, \mathrm{e}^{-\mathrm{i}(E_n^2 - E_m^2)\sum_k (g_k^2 \sin(\omega_k t)/\hbar^2\omega_k^2)} \\ & \times \exp\left[-\frac{1}{2}(E_m - E_n)^2 \sum_k \frac{g_k^2}{\hbar^2\omega_k^2}\coth\left(\frac{\beta\hbar\omega_k}{2}\right)\right. \\ & \left.\times\, |(\mathrm{e}^{\mathrm{i}\omega_k t} - 1)\cosh(r_k) + (\mathrm{e}^{-\mathrm{i}\omega_k t} - 1)\sinh(r_k)\,\mathrm{e}^{2\mathrm{i}\Phi_k}|^2\right]\rho^s_{nm}(0). \end{aligned} \tag{9}$$

Differentiating equation (9) with respect to time, we obtain the master equation giving the system evolution under the influence of the environment as

$$\dot{\rho}^s_{nm}(t) = \left[-\frac{\mathrm{i}}{\hbar}(E_n - E_m) + \mathrm{i}\dot{\eta}(t)\left(E_n^2 - E_m^2\right) - (E_n - E_m)^2\dot{\gamma}(t)\right]\rho^s_{nm}(t), \tag{10}$$

where

$$\eta(t) = -\sum_k \frac{g_k^2}{\hbar^2\omega_k^2}\sin(\omega_k t), \tag{11}$$

and

$$\gamma(t) = \frac{1}{2}\sum_k \frac{g_k^2}{\hbar^2\omega_k^2}\coth\left(\frac{\beta\hbar\omega_k}{2}\right)|(\mathrm{e}^{\mathrm{i}\omega_k t} - 1)\cosh(r_k) + (\mathrm{e}^{-\mathrm{i}\omega_k t} - 1)\sinh(r_k)\,\mathrm{e}^{2\mathrm{i}\Phi_k}|^2. \tag{12}$$

For the case of zero squeezing, $r = \Phi = 0$, and $\gamma(t)$ given by equation (12) reduces to the expression obtained earlier [9–11] for the case of a thermal bath. It can be seen that $\eta(t)$ (11) is independent of the bath initial conditions and hence remains the same as for the thermal bath. Comparing the master equation obtained for the case of a QND coupling to the bath (10) with the master equation obtained in the case of QBM as in [26–28], where the master equation was obtained for the QBM of the system of a harmonic oscillator in a squeezed thermal bath, we find that the term responsible for decoherence in the QND case is given by $\dot{\gamma}(t)$. It is interesting to note that in contrast to the QBM case, here there is no term governing dissipation. Also missing are the various other diffusion terms, namely, those responsible for promoting diffusion in p^2 and those responsible for diffusion in $xp + px$, the so-called anomalous diffusion terms. Also note that in the exponent of the third exponential on the right-hand side of equation (9), responsible for the decay of coherences, the coefficient of $\gamma(t)$ is dependent on the eigenvalues E_n of the 'conserved pointer observable' operator which in this case is the system Hamiltonian itself. This reiterates the observation that the decay of coherence in a system interacting with its bath via a QND interaction depends on the conserved pointer observable and the bath coupling parameters [10].

2.2. Bath of two-level systems

We briefly take up the case of a bath of two-level systems to illustrate in a transparent manner its difference with a bath of harmonic oscillators. The Hamiltonian considered is

$$\begin{aligned} H &= H_S + H_R + H_{SR} \\ &= H_S + \sum_k \omega_k \sigma_{zk} + H_S \sum_k C_k \sigma_{xk}. \end{aligned} \tag{13}$$

Since $[H_S, H_{SR}] = 0$, the Hamiltonian (13) is of a QND type. Starting from the unitary evolution of the entire closed system and then tracing over the bath variables, we obtain the reduced density matrix in the system eigenbasis as

$$\rho^s_{nm}(t) = \mathrm{e}^{-\mathrm{i}(E_n - E_m)t/\hbar} \, \mathrm{Tr}_{\mathrm{R}}[\mathrm{e}^{\mathrm{i}H_m t/\hbar} \, \mathrm{e}^{-\mathrm{i}H_n t/\hbar} \rho_R(0)] \rho^s_{nm}(0), \tag{14}$$

where

$$H_n = \sum_k [\omega_k \sigma_{zk} + E_n C_k \sigma_{xk}] = \sum_k O_k(E_n). \tag{15}$$

Using the properties of the σ_z, σ_x matrices, it can be seen that

$$\mathrm{e}^{\mathrm{i}O_k(E_m)t} = \cos(\omega'_k(E_m)t) + \frac{\mathrm{i}\sin(\omega'_k(E_m)t)}{\omega'_k(E_m)} (\omega_k \sigma_{zk} + E_m C_k \sigma_{xk}), \tag{16}$$

where

$$\omega'_k(E_m) = \sqrt{\omega_k^2 + E_m^2 C_k^2}. \tag{17}$$

Thus,

$$\begin{aligned} \mathrm{e}^{\mathrm{i}O_k(E_m)t} \, \mathrm{e}^{-\mathrm{i}O_k(E_n)t} &= \cos(\omega'_k(E_m)t)\cos(\omega'_k(E_n)t) \\ &+ \frac{\sin(\omega'_k(E_m)t)\sin(\omega'_k(E_n)t)}{\omega'_k(E_m)\omega'_k(E_n)} \left(\omega_k^2 + E_m E_n C_k^2\right) \\ &+ \frac{\sin(\omega'_k(E_m)t)\sin(\omega'_k(E_n)t)}{\omega'_k(E_m)\omega'_k(E_n)} \omega_k C_k (E_n - E_m) \sigma_{zk}\sigma_{xk} \\ &- \frac{\mathrm{i}\cos(\omega'_k(E_m)t)\sin(\omega'_k(E_n)t)}{\omega'_k(E_n)} (\omega_k \sigma_{zk} + E_n C_k \sigma_{xk}) \\ &+ \frac{\mathrm{i}\cos(\omega'_k(E_n)t)\sin(\omega'_k(E_m)t)}{\omega'_k(E_m)} (\omega_k \sigma_{zk} + E_m C_k \sigma_{xk}). \end{aligned} \tag{18}$$

Using (18) in (14), it can be seen that only the first two terms on the right-hand side of equation (18) contribute and the reduced density matrix of the system becomes

$$\begin{aligned} \rho^s_{nm}(t) = \mathrm{e}^{-\mathrm{i}(E_n - E_m)t/\hbar} \prod_k \Bigg[& \cos(\omega'_k(E_m)t)\cos(\omega'_k(E_n)t) \\ &+ \frac{\sin(\omega'_k(E_m)t)\sin(\omega'_k(E_n)t)}{\omega'_k(E_m)\omega'_k(E_n)} \left(\omega_k^2 + E_m E_n C_k^2\right) \Bigg] \rho^s_{nm}(0), \end{aligned} \tag{19}$$

as also obtained by Shao *et al* [9]. We can see from equation (19) that the reduced density matrix of the system is independent of the temperature and squeezing conditions of the reservoir, as may be expected from the structure of the Hamiltonian (13). This brings out the intrinsic difference between a bosonic bath of harmonic oscillators and a bath of two-level systems.

3. Decoherence dynamics

Here onwards we consider only the bosonic bath of harmonic oscillators in an initial squeezed thermal state. In this section, we analyze the decay of coherence of our generic system under various environmental conditions. We examine the decoherence term in the master equation for the reduced density matrix of the system. We also compute the dynamics of the 'measure of coherence' related to the linear entropy of the system for the zero- as well as the high-temperature cases for different degrees of squeezing of the bath.

3.1. Dephasing in the system master equation

In this subsection, we examine in detail the term $\gamma(t)$ (equation (12)). This is the term whose time derivative is the decoherence-causing term as is evident from the master equation (10). To proceed, we assume a 'quasi-continuous' bath spectrum with spectral density $I(\omega)$ such that

$$\sum_k \frac{g_k^2}{\hbar^2} f(\omega_k) \longrightarrow \int_0^\infty \mathrm{d}\omega\, I(\omega) f(\omega), \tag{20}$$

and using an Ohmic spectral density

$$I(\omega) = \frac{\gamma_0}{\pi} \omega\, \mathrm{e}^{-\omega/\omega_c}, \tag{21}$$

where γ_0 and ω_c are two bath parameters, we obtain $\eta(t)$ in (11) as

$$\eta(t) = -\frac{\gamma_0}{\pi} \tan^{-1}(\omega_c t). \tag{22}$$

In the limit $\omega_c t \gg 1$, $\tan^{-1}(\omega_c t) \longrightarrow \frac{\pi}{2}$ and $\eta(t) \longrightarrow -\frac{\gamma_0}{2}$. Now we evaluate $\gamma(t)$ given in (12) for the squeezed thermal bath for the cases of zero-T and high-T.

$T = 0$. Using equations (20) and (21) in equation (12) and using the zero-T limit, we obtain $\gamma(t)$ as

$$\gamma(t) = \frac{\gamma_0}{2\pi} \cosh(2r) \ln\left(1 + \omega_c^2 t^2\right) - \frac{\gamma_0}{4\pi} \sinh(2r) \ln\left[\frac{\left(1 + 4\omega_c^2 (t-a)^2\right)}{\left(1 + \omega_c^2 (t-2a)^2\right)^2}\right] - \frac{\gamma_0}{4\pi} \sinh(2r) \ln\left(1 + 4a^2\omega_c^2\right), \tag{23}$$

where $t > 2a$. Here we have taken, for simplicity, the squeezed bath parameters as

$$\cosh(2r(\omega)) = \cosh(2r), \qquad \sinh(2r(\omega)) = \sinh(2r), \qquad \Phi(\omega) = a\omega, \tag{24}$$

where a is a constant depending upon the squeezed bath. The decoherence-causing term $\dot{\gamma}(t)$ is obtained from the above equation as

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = \frac{\gamma_0}{\pi} \cosh(2r) \frac{\omega_c^2 t}{\left(1 + \omega_c^2 t^2\right)} - \frac{\gamma_0}{4\pi} \sinh(2r) \left[\frac{8\omega_c^2 (t-a)}{1 + 4\omega_c^2 (t-a)^2} - \frac{4\omega_c^2 (t-2a)}{1 + \omega_c^2 (t-2a)^2}\right]. \tag{25}$$

We can see from the above equation that in the long time limit, $\dot{\gamma}(t) \longrightarrow \gamma_0 \cosh(2r)/(\pi t)$, and the terms proportional to the sine hyperbolic function, coming from the nonstationarity of the squeezed bath, are washed out. For the case of zero squeezing, we obtain from (23)

$$\gamma(t) = \frac{\gamma_0}{2\pi} \ln\left(1 + \omega_c^2 t^2\right) \longrightarrow \frac{\gamma_0}{2\pi} \times \text{constant}, \tag{26}$$

because of the slow logarithmic behavior.

As $\omega_c \longrightarrow \infty$, $\gamma(t)$ in (23) tends to

$$\gamma(t) \longrightarrow \frac{\gamma_0}{2\pi} \cosh(2r) A - \frac{\gamma_0}{4\pi} \sinh(2r) B, \tag{27}$$

where $A = \lim_{\omega_c \longrightarrow \infty} \ln\left(1 + \omega_c^2 t^2\right) = \text{constant}$, because of the slow logarithmic behavior, and $B = \lim_{\omega_c \longrightarrow \infty} \ln\left[\frac{\left(1+4\omega_c^2(t-a)^2\right)\left(1+4a^2\omega_c^2\right)}{\left(1+\omega_c^2(t-2a)^2\right)^2}\right] = \text{constant}$, again because of the slow logarithmic behavior.

High T. Using (20) and (21) in (12) and using the high-*T* limit, we obtain

$$\begin{aligned}
\gamma(t) = {} & \frac{\gamma_0 k_B T}{\pi \hbar \omega_c} \cosh(2r) \left[2\omega_c t \tan^{-1}(\omega_c t) + \ln\left(\frac{1}{1 + \omega_c^2 t^2}\right)\right] \\
& - \frac{\gamma_0 k_B T}{2\pi \hbar \omega_c} \sinh(2r) \Bigg[4\omega_c (t-a) \tan^{-1}(2\omega_c (t-a)) \\
& - 4\omega_c (t-2a) \tan^{-1}(\omega_c (t-2a)) + 4a\omega_c \tan^{-1}(2a\omega_c) \\
& + \ln\left(\frac{\left[1 + \omega_c^2 (t-2a)^2\right]^2}{\left[1 + 4\omega_c^2 (t-a)^2\right]}\right) + \ln\left(\frac{1}{1 + 4a^2\omega_c^2}\right)\Bigg],
\end{aligned} \tag{28}$$

where $t > 2a$. From (28) we can obtain $\gamma(t)$ for high T and thermal bath with no squeezing, by setting r and a to zero, as

$$\gamma(t) = \frac{\gamma_0 k_B T}{\pi \hbar \omega_c} \left[2\omega_c t \tan^{-1}(\omega_c t) + \ln\left(\frac{1}{1 + \omega_c^2 t^2}\right)\right], \tag{29}$$

such that

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = \frac{2\gamma_0 k_B T}{\pi \hbar} \tan^{-1}(\omega_c t). \tag{30}$$

This matches with the result obtained in [11]. For the case of a squeezed thermal bath, we can obtain $\dot{\gamma}(t)$ from (28) as

$$\begin{aligned}
\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = {} & \frac{2\gamma_0 k_B T}{\pi \hbar} \cosh(2r) \tan^{-1}(\omega_c t) - \frac{2\gamma_0 k_B T}{\pi \hbar} \sinh(2r) \\
& \times [\tan^{-1}(2\omega_c (t-a)) - \tan^{-1}(\omega_c (t-2a))].
\end{aligned} \tag{31}$$

Figure 1 depicts the behavior of the decoherence-causing term, $\dot{\gamma}(t)$ (equation (25)), for $T = 0$ while figure 2 depicts its behavior for high-*T* (equation (31)), with and without bath squeezing indicated by the parameter r. A comparison between the two clearly indicates the power-law behavior of the decay of coherences at $T = 0$ and an exponential decay at high T.

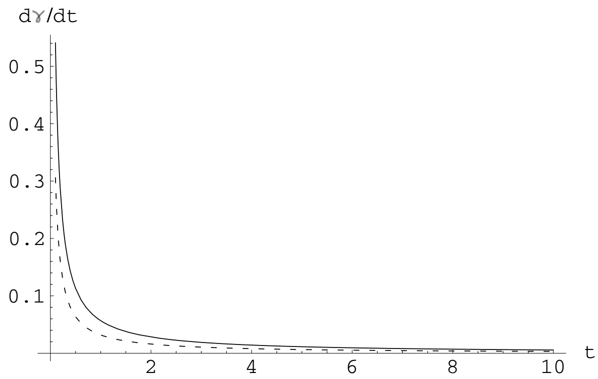

Figure 1. $\frac{d\gamma(t)}{dt}$ (equation (25)) as a function of time t for different environmental conditions. Here, $\gamma_0 = 0.1$, $\omega_c = 50$, $a = 0$ and temperature $T = 0$. The dashed and solid curves correspond to the environmental squeezing parameter (equation (24)) $r = 0$ and 0.4, respectively.

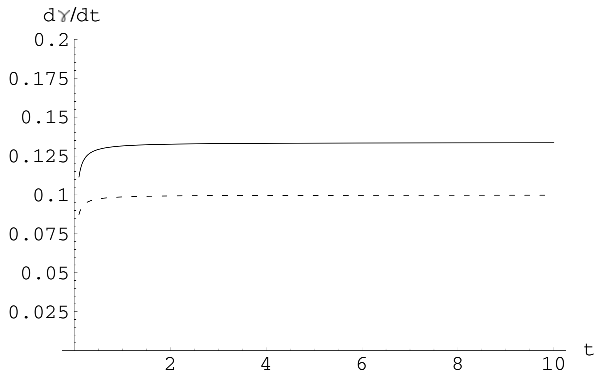

Figure 2. $\frac{d\gamma(t)}{dt}$ (equation (31)) as a function of time t for different environmental conditions. Here, $\gamma_0 = 0.1$, $\omega_c = 50$, $a = 0$ and temperature T (in units where $\hbar \equiv k_B \equiv 1$) = 300. The dashed and solid curves correspond to the environmental squeezing parameter (equation (24)) $r = 0$ and 0.4, respectively.

As $\omega_c \longrightarrow \infty$, from (28) we get

$$\gamma(t) \longrightarrow \frac{\gamma_0 k_B T}{\hbar} \cosh(2r)t - 2\frac{\gamma_0 k_B T}{\hbar} \sinh(2r)a. \tag{32}$$

3.2. Evolution of the linear entropy—measure of coherence

It is well known that for a pure state $\rho^2(t) = \rho(t)$ and $\mathrm{Tr}[\rho^2(t)] = 1$. A mixed state instead is defined by the class of states which satisfies the inequality $\mathrm{Tr}[\rho^2(t)] \leqslant 1$. This leads to the definition of the linear entropy of a quantum state, $S(t) \equiv 1 - \mathrm{Tr}[\rho^2(t)]$, which is positive: $S(t) \geqslant 0$, and bounded: $S(t) \leqslant 1$. $S(t) = 0$ for a pure state and 1 for a completely mixed state. We can set up a related 'measure of coherence' of the system following [9] as

$$C(t) \equiv \mathrm{Tr}[\rho^s(t)]^2. \tag{33}$$

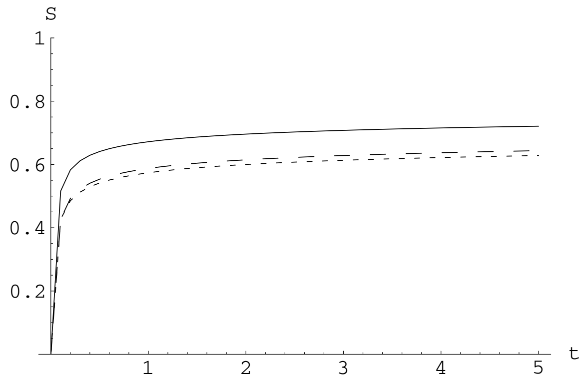

Figure 3. Linear entropy $S(t)$ (equation (36)) as a function of time t for different environmental conditions: $\gamma_0 = 0.1, \omega = 1, \omega_c = 50, a = 0, |\alpha|^2 = 5$ and $T = 0$ so that equation (23) is used. The large-dashed, small-dashed and solid curves correspond to the environmental squeezing parameter (equation (24)) $r = 0, -0.3$ and 0.4, respectively.

If we assume the system to start from a pure state,

$$\rho^s(0) = \left[\sum_n p_n|n\rangle\right]\left[\sum_m p_m^*\langle m|\right], \tag{34}$$

then using (9) we have

$$C(t) = \sum_{m,n} |p_n|^2|p_m|^2\, \mathrm{e}^{-2(E_n-E_m)^2\gamma(t)}, \tag{35}$$

where $\gamma(t)$ is as in (12).

The linear entropy $S(t)$ can be computed from $C(t)$ as

$$S(t) = \mathrm{Tr}[\rho^s(t) - (\rho^s(t))^2] = \mathcal{I} - C(t). \tag{36}$$

$S(t)$ is plotted in figures 3 and 4 for a harmonic oscillator system starting out in a coherent state $|\alpha\rangle$ [31], for temperatures $T = 0$ and 300, respectively, and for various values of environmental squeezing parameter r.

$T = 0$. Using equation (12) in equation (35) and applying the $T = 0$ limit, i.e., making use of (23), the measure of coherence is obtained as

$$C(t) = \sum_{n,m} |p_n|^2|p_m|^2\left(1+\omega_c^2t^2\right)^{-\gamma_0\cosh(2r)(E_n-E_m)^2/\pi} \times \left[\frac{\left(1+4\omega_c^2(t-a)^2\right)}{\left(1+\omega_c^2(t-2a)^2\right)^2}\right]^{\gamma_0\sinh(2r)(E_n-E_m)^2/(2\pi)} \times \left(1+4a^2\omega_c^2\right)^{\gamma_0\sinh(2r)(E_n-E_m)^2/(2\pi)}. \tag{37}$$

For the case of zero squeezing, $r = 0 = a$, and $C(t)$ given by equation (37) becomes

$$C(t) = \sum_{n,m} |p_n|^2|p_m|^2(\omega_c t)^{-2\gamma_0(E_n-E_m)^2/\pi}. \tag{38}$$

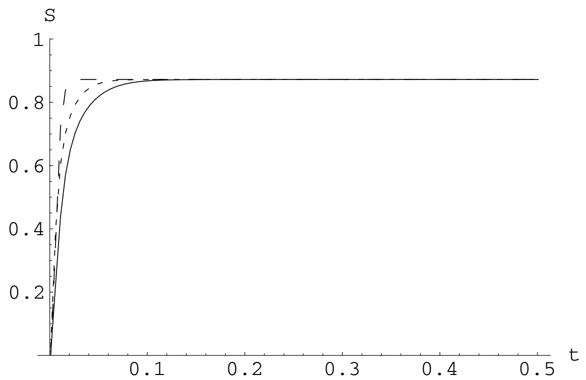

Figure 4. Linear entropy $S(t)$ (equation (36)) as a function of time t for different environmental conditions: $\gamma_0 = 0.1$, $\omega = 1$, $\omega_c = 50$, $a = 0$, $|\alpha|^2 = 5$ and T (in units where $\hbar \equiv k_B \equiv 1$) $= 300$ so that equation (28) is used. The solid, small-dashed and large-dashed curves correspond to the environmental squeezing parameter (equation (24)) $r = 0$, -0.5 and 2, respectively.

Here we have in addition imposed the condition $\omega_c t \gg 1$, which is a valid experimentally accessible domain of time. This agrees with the result obtained in [9]. It can be seen from equations (37) and (38) that coherences follow the 'power law' for $T = 0$.

High T. Using equation (12) in equation (35) and applying the high-T limit, i.e., using (28), the measure of coherence is obtained as

$$\begin{aligned} C(t) = \sum_{m,n} |p_n|^2 |p_m|^2 \exp\Bigg\{ &-(E_n - E_m)^2 \frac{4\gamma_0 k_B T}{\pi\hbar} \\ &\times [\cosh(2r)\tan^{-1}(\omega_c t) - \sinh(2r)\tan^{-1}(2\omega_c(t-a)) \\ &+ \sinh(2r)\tan^{-1}(\omega_c(t-2a))]t - (E_n - E_m)^2 \frac{4a\gamma_0 k_B T}{\pi\hbar} \\ &\times \sinh(2r)[\tan^{-1}(2\omega_c(t-a)) - 2\tan^{-1}(\omega_c(t-2a)) - \tan^{-1}(2a\omega_c)] \Bigg\} \\ &\times \left(1 + \omega_c^2 t^2\right)^{2\gamma_0 k_B T \cosh(2r)(E_n - E_m)^2/(\pi\hbar\omega_c)} \\ &\times \left(\frac{\left[1 + \omega_c^2 (t-2a)^2\right]^2}{\left[1 + 4\omega_c^2 (t-a)^2\right]} \right)^{\gamma_0 k_B T \sinh(2r)(E_n - E_m)^2/(\pi\hbar\omega_c)} \\ &\times \left(1 + 4a^2\omega_c^2\right)^{-\gamma_0 k_B T \sinh(2r)(E_n - E_m)^2/(\pi\hbar\omega_c)}. \end{aligned} \qquad (39)$$

It can be seen from (39) that in the high-T case, the measure of coherence involves exponential as well as power-law terms. It is also evident that the terms dominating the temporal behavior of the coherence measure $C(t)$ are

$$\begin{aligned} \sum_{n,m} |p_n|^2 |p_m|^2 \exp\Bigg\{ &-(E_n - E_m)^2 \frac{4\gamma_0 k_B T}{\pi\hbar} [\cosh(2r)\tan^{-1}(\omega_c t) \\ &- \sinh(2r)\tan^{-1}(2\omega_c(t-a)) + \sinh(2r)\tan^{-1}(\omega_c(t-2a))]t \Bigg\}. \end{aligned}$$

Thus, in the high-T limit, the behavior of the coherences is predominantly exponential. In the long time limit ($\omega_c t \to \infty$),

$$C(t) \to \sum_{n,m} |p_n|^2 |p_m|^2 \exp\left\{-(E_n - E_m)^2 \frac{2\gamma_0 k_B T}{\hbar} \cosh(2r) t\right\}. \tag{40}$$

By comparing figures 3 and 4, it is evident that at $T = 0$ (figure 3), the coherences stay for a longer time characterizing the power-law decay as opposed to the high-T case (figure 4), where the exponential decay causes the coherences to diminish over a much shorter period of time. Also evident is the effect of bath squeezing, characterized by the parameter r, on the coherences in the two temperature regimes. While in the zero-T case, the effect of squeezing remains over a longer period of time, in the high-T case it diminishes quickly. In this, its behavior is similar to that of the QBM of a harmonic oscillator system [27, 28] at high T. Another interesting feature that comes out is that in the zero-T regime (figure 3), by suitably adjusting the bath squeezing parameter r, the coherence in the system can be improved over the unsqueezed bath, as seen by comparing the small-dashed curve with the large-dashed one, representing the bath squeezing parameter (24) $r = -0.3$ and 0, respectively. This clearly brings out the utility of squeezing of the thermal bath.

4. Comparison of the QND and non-QND evolutions and phase diffusion in QND

In this section, we make a comparison between the processes underlying the QND and non-QND (i.e., where $[H_S, H_{SR}] \neq 0$) types of evolution for a two-level atomic system and a harmonic oscillator. We briefly consider the question of phase diffusion in the QND evolution of a harmonic oscillator.

4.1. Two-level system

Here we take the system to be a two-level atomic system, with the Hamiltonian

$$H_S = \frac{\hbar\omega}{2}\sigma_z, \tag{41}$$

σ_z being the usual Pauli matrix, to be substituted in equation (1). This is a common system, with a lot of recent applications, as for example, in the quantum computation models in [32–34].

4.1.1. QND evolution. In order to study the reduced density matrix of the system under a QND system–reservoir interaction, i.e., for equation (9), we need to identify an appropriate system eigenbasis. Here this is provided by the Wigner–Dicke states [35–37] $|j, m\rangle$, which are the simultaneous eigenstates of the angular momentum operators J^2 and J_z, and we have

$$H_S |j, m\rangle = \hbar\omega m |j, m\rangle = E_{j,m} |j, m\rangle, \tag{42}$$

where $-j \leqslant m \leqslant j$. For the two-level system considered here, $j = \frac{1}{2}$ and hence $m = -\frac{1}{2}, \frac{1}{2}$. Using this in equation (9) and starting the system from the state

$$|\psi(0)\rangle = \cos\left(\frac{\theta_0}{2}\right)|1\rangle + \mathrm{e}^{\mathrm{i}\phi_0} \sin\left(\frac{\theta_0}{2}\right)|0\rangle, \tag{43}$$

the reduced density matrix of the system after time t is [25]

$$\rho^s_{m,n}(t) = \begin{pmatrix} \cos^2\left(\frac{\theta_0}{2}\right) & \frac{1}{2}\sin(\theta_0)\,\mathrm{e}^{-\mathrm{i}(\omega t+\phi_0)}\,\mathrm{e}^{-(\hbar\omega)^2\gamma(t)} \\ \frac{1}{2}\sin(\theta_0)\,\mathrm{e}^{\mathrm{i}(\omega t+\phi_0)}\,\mathrm{e}^{-(\hbar\omega)^2\gamma(t)} & \sin^2\left(\frac{\theta_0}{2}\right) \end{pmatrix}, \tag{44}$$

from which the Bloch vectors can be extracted to yield

$$\begin{aligned}\langle\sigma_x(t)\rangle &= \sin(\theta_0)\cos(\omega t+\phi_0)\,\mathrm{e}^{-(\hbar\omega)^2\gamma(t)},\\ \langle\sigma_y(t)\rangle &= \sin(\theta_0)\sin(\omega t+\phi_0)\,\mathrm{e}^{-(\hbar\omega)^2\gamma(t)},\\ \langle\sigma_z(t)\rangle &= \cos(\theta_0).\end{aligned}\tag{45}$$

Here, $\gamma(t)$ is as in equations (23) and (28) for zero and high T, respectively, and $\sigma_x, \sigma_y, \sigma_z$ are the standard Pauli matrices. It can be easily seen from the above Bloch vector equations that the QND evolution causes a coplanar, fixed by the polar angle θ_0, in-spiral towards the z-axis of the Bloch sphere. This is the characteristic of a phase-damping channel [38].

4.1.2. Non-QND evolution of the Lindblad form. Next we study the reduced dynamics of the system (41) interacting with a squeezed thermal bath under a weak Born–Markov and rotating wave approximation. This implies that here the system interacts with its environment via a non-QND interaction such that along with a loss in phase information, energy dissipation also takes place. The evolution has a Lindblad form which in the interaction picture is given by [29, 31]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho^s(t) = \gamma_0(N+1)&\left(\sigma_-\rho^s(t)\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho^s(t) - \frac{1}{2}\rho^s(t)\sigma_+\sigma_-\right)\\ &+\gamma_0 N\left(\sigma_+\rho^s(t)\sigma_- - \frac{1}{2}\sigma_-\sigma_+\rho^s(t) - \frac{1}{2}\rho^s(t)\sigma_-\sigma_+\right)\\ &-\gamma_0 M\sigma_+\rho^s(t)\sigma_+ - \gamma_0 M^*\sigma_-\rho^s(t)\sigma_-.\end{aligned}\tag{46}$$

Here,

$$N = N_{\mathrm{th}}(\cosh^2 r + \sinh^2 r) + \sinh^2 r,\tag{47}$$

$$M = -\tfrac{1}{2}\sinh(2r)\,\mathrm{e}^{\mathrm{i}\Phi}(2N_{\mathrm{th}}+1),\tag{48}$$

and

$$N_{\mathrm{th}} = \frac{1}{\mathrm{e}^{\hbar\omega/(k_B T)}-1},\tag{49}$$

where N_{th} is the Planck distribution giving the number of thermal photons at frequency ω, and r, Φ are squeezing parameters of the bath. The case of a thermal bath without squeezing can be obtained from the above expressions by setting these squeezing parameters to zero. γ_0 is a constant typically denoting the system–environment coupling strength, and σ_+, σ_- are the standard raising and lowering operators, respectively, given by

$$\begin{aligned}\sigma_+ &= |1\rangle\langle 0| = \tfrac{1}{2}(\sigma_x + \mathrm{i}\sigma_y),\\ \sigma_- &= |0\rangle\langle 1| = \tfrac{1}{2}(\sigma_x - \mathrm{i}\sigma_y).\end{aligned}\tag{50}$$

In the above equation, $|1\rangle$ is the upper state of the atom and $|0\rangle$ is the lower state. Evolving the system given by H_S from the initial state given in equation (43), using equation (46), we obtain the reduced density matrix of the system from which the Bloch vectors can be extracted to yield [25]

$$\begin{aligned}\langle\sigma_x(t)\rangle = &\left[1+\frac{1}{2}(\mathrm{e}^{\gamma_0 a t}-1)(1+\cos\Phi)\right]\mathrm{e}^{-\gamma_0(2N+1+a)t/2}\langle\sigma_x(0)\rangle\\ &-\sin\Phi\sinh\left(\frac{\gamma_0 a t}{2}\right)\mathrm{e}^{-\gamma_0(2N+1)t/2}\langle\sigma_y(0)\rangle,\end{aligned}$$

$$\langle\sigma_y(t)\rangle = \left[1 + \frac{1}{2}(\mathrm{e}^{\gamma_0 a t} - 1)(1 - \cos\Phi)\right]\mathrm{e}^{-\gamma_0(2N+1+a)t/2}\langle\sigma_y(0)\rangle$$
$$- \sin\Phi \sinh\left(\frac{\gamma_0 a t}{2}\right)\mathrm{e}^{-\gamma_0(2N+1)t/2}\langle\sigma_x(0)\rangle,$$
$$\langle\sigma_z(t)\rangle = \mathrm{e}^{-\gamma_0(2N+1)t}\langle\sigma_z(0)\rangle - \frac{1}{(2N+1)}(1 - \mathrm{e}^{-\gamma_0(2N+1)t}), \tag{51}$$

where

$$a = \sinh(2r)(2N_{\text{th}} + 1). \tag{52}$$

It can be seen from equation (51) that the reduced density matrix $\rho^s(t)$ shrinks towards the asymptotic equilibrium state ρ_{asymp}, given by

$$\rho_{\text{asymp}} = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}, \tag{53}$$

where $p = \frac{1}{2}\left[1 + \frac{1}{(2N+1)}\right]$. For the case of zero squeezing and zero temperature, this action corresponds to an amplitude-damping channel [25, 38] with the Bloch sphere shrinking to a point representing the state $|0\rangle$ (the south pole of the Bloch sphere) while for the case of finite T but zero squeezing, the above action corresponds to a generalized amplitude-damping channel [25, 38] with the Bloch sphere shrinking to a point along the line joining the south pole to the center of the Bloch sphere. The center of the Bloch sphere is reached in the limit of infinite temperature.

The above analysis brings out the point that while the case of the QND system–environment interaction corresponds to a phase-damping channel, the case where the evolution is non-QND, in particular where the evolution is generated by equation (46), having a Lindblad form, corresponds to a (generalized) amplitude-damping channel (for zero bath-squeezing). This brings out in a very transparent manner the difference in the quantum statistical mechanics underlying the two processes. While in the case of QND interaction, the system tends (along the z-axis) towards a localized state, for the case of non-QND interaction, the system tends towards a unique asymptotic equilibrium state, which would be pure (for $T = 0$) or mixed (for $T > 0$). This can be seen from figure 5, where the effect of the environment on the initial Bloch sphere (figure 5(A)) is brought out. Figure 5(B) depicts the evolution under a QND system–environment interaction (equation (45)) while figures 5(C) and (D) depict the evolution under a dissipative system–environment interaction (equation (51)). While figure 5(B) clearly shows a tendency of localization along the z-axis, figures 5(C) and (D) illustrate the tendency of going towards a unique asymptotic fixed point. In figure 5(D), the presence of a finite Φ (48) is manifested in the tilt in the figure.

4.2. *Harmonic oscillator system*

Next, we take a system of harmonic oscillator with the Hamiltonian

$$H_S = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right). \tag{54}$$

The Hamiltonian H_S (54), substituted in equation (1), has been used by Turchette *et al* [39] to describe an experimental study of the decoherence and decay of quantum states of a trapped atomic ion's harmonic motion interacting with an engineered 'phase reservoir'.

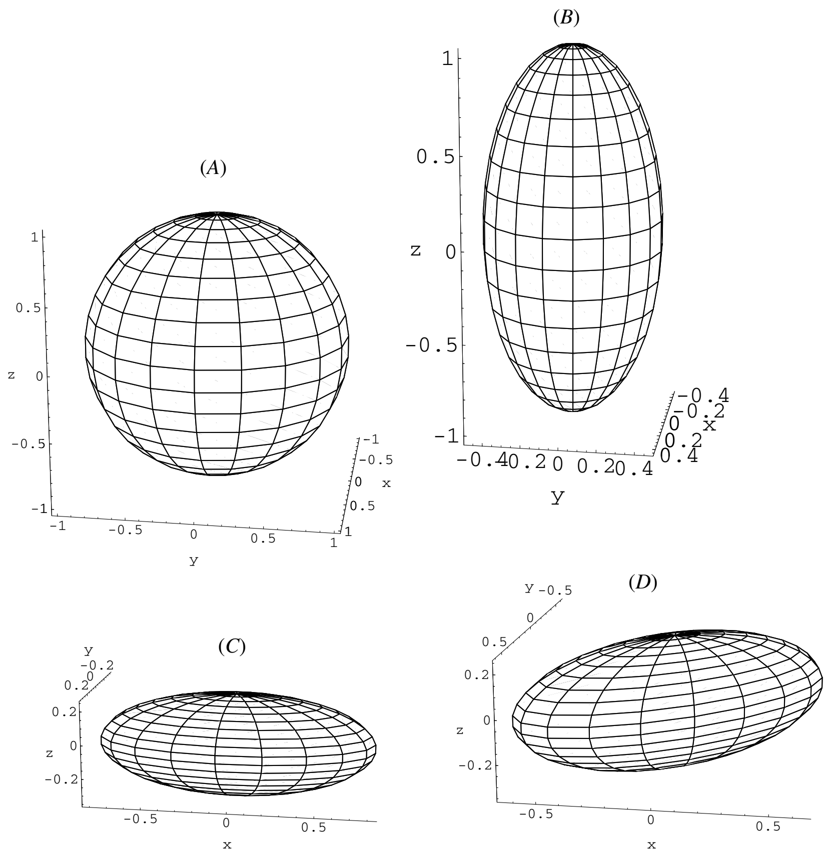

Figure 5. Effect of the QND and dissipative interactions on the Bloch sphere: (*A*) the full Bloch sphere; (*B*) the Bloch sphere after time $t = 20$, with $\gamma_0 = 0.2, T = 0, \omega = 1, \omega_c = 40\omega$ and the environmental squeezing parameter (equation (24)) $r = a = 0.5$, evolved under a QND interaction (equation (45)); (*C*) and (*D*) the effect of the Born–Markov type of dissipative interaction (equation (51)) with $\gamma_0 = 0.6$ and temperature $T = 5$, on the Bloch sphere—the *x*- and *y*-axes are interchanged to present the effect of squeezing more clearly. (*C*) corresponds to $r = 0.4, \Phi = 0$ and $t = 0.15$ while (*D*) corresponds to $r = 0.4, \Phi = 1.5$ and $t = 0.15$.

4.2.1. QND evolution. Noting that the master equation (10) in the system space can be written equivalently as

$$\dot{\rho}^s = -\frac{\mathrm{i}}{\hbar}[H_S, \rho^s] + \mathrm{i}\dot{\eta}(t)\big[H_S^2, \rho^s\big] - \dot{\gamma}(t)\big(H_S^2\rho^s - 2H_S\rho^s H_S + \rho^s H_S^2\big), \quad (55)$$

and substituting (54) in (55) we obtain the master equation for a harmonic oscillator coupled to a bosonic bath of harmonic oscillators by a QND type of coupling as

$$\dot{\rho}^s = -\mathrm{i}\omega[a^\dagger a, \rho^s] + \mathrm{i}\hbar^2\omega^2\dot{\eta}(t)[(a^\dagger a)^2 + a^\dagger a, \rho^s] \\ - \hbar^2\omega^2\dot{\gamma}(t)[(a^\dagger a)^2\rho^s - 2a^\dagger a\rho^s a^\dagger a + \rho^s(a^\dagger a)^2]. \quad (56)$$

For clarity we transform the above equation into the form of a Q distribution function [31] given by the prescription

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle, \quad (57)$$

where $|\alpha\rangle$ is a coherent state. From the master equation (56), the equation for the Q distribution function becomes

$$\frac{\partial}{\partial t}Q = -\mathrm{i}\omega\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)Q + \mathrm{i}\hbar^2\omega^2\dot{\eta}(t)\Bigg[2(1+\alpha\alpha^*)\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right) + \left(\alpha^{*2}\frac{\partial^2}{\partial\alpha^{*2}} - \alpha^2\frac{\partial^2}{\partial\alpha^2}\right)\Bigg]Q - \hbar^2\omega^2\dot{\gamma}(t) \times\left[\alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} + \alpha^{*2}\frac{\partial^2}{\partial\alpha^{*2}} + \alpha^2\frac{\partial^2}{\partial\alpha^2} - 2\alpha\alpha^*\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]Q. \quad (58)$$

Using polar coordinates, $\alpha = \xi\,\mathrm{e}^{\mathrm{i}\theta}$, this equation can be transformed into

$$\frac{\partial}{\partial t}Q = \omega\frac{\partial}{\partial\theta}Q - \hbar^2\omega^2\dot{\eta}(t)\left[(1+2\xi^2)\frac{\partial}{\partial\theta} + \xi\frac{\partial^2}{\partial\xi\partial\theta}\right]Q + \hbar^2\omega^2\dot{\gamma}(t)\frac{\partial^2}{\partial\theta^2}Q. \quad (59)$$

4.2.2. Non-QND QBM. To compare (58) with the equation obtained in the case of QBM of a system of harmonic oscillator interacting with a squeezed thermal bath, we start with the general QBM master equation [27, 28]:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\rho^s(x,x',t) = \left\{\frac{-\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right) + \frac{M}{2}\Omega^2_{\mathrm{ren}}(t)(x^2 - x'^2)\right\}\rho^s(x,x',t) - \mathrm{i}\hbar\Gamma(t)(x-x')\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\rho^s(x,x',t) + \mathrm{i}D_{pp}(t)(x-x')^2\rho^s(x,x',t) - \hbar(D_{xp}(t) + D_{px}(t))(x-x')\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right)\rho^s(x,x',t) - \mathrm{i}\hbar^2 D_{xx}(t)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right)^2\rho^s(x,x',t). \quad (60)$$

Here, $\Gamma(t)$ is the term responsible for dissipation, $D_{pp}(t)$ for decoherence, $D_{xx}(t)$ promotes diffusion in p^2, and $D_{xp}(t)$, $D_{px}(t)$ are responsible for promoting (anomalous) diffusion in $xp + px$. The details of these coefficients of the master equation (60) can be found in [27, 28]. Here, the coordinate representation of the density matrix has been used in contrast to the energy representation used in (10). Comparing (60) with (10) we find that the QND coupling of the system with the environment makes the quantum statistical mechanics of the evolution much simpler. As already noted below equation (12), a comparison between (10) and (60) shows that in the QND case there is a decoherence-governing term $\dot{\gamma}(t)$, but no term responsible for dissipation. In contrast, the QBM case has dissipation and a number of *diffusion channels* as seen by the existence of the diffusion terms $D_{xx}(t)$, $D_{xp}(t) + D_{px}(t)$ and $D_{pp}(t)$.

Since equation (60) is also obtained for a harmonic oscillator system (cf (54)), we proceed as before and obtain its corresponding Q equation as

$$\mathrm{i}\hbar\frac{\partial}{\partial t}Q = \hbar\omega\left[\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right]Q + \frac{\mathrm{i}\hbar}{2}\Gamma(t)\Bigg[-\frac{\partial^2}{\partial\alpha^{*2}} - \frac{\partial^2}{\partial\alpha^2} + 2\frac{\partial^2}{\partial\alpha\partial\alpha^*} - 2\alpha\frac{\partial}{\partial\alpha^*} - 2\alpha^*\frac{\partial}{\partial\alpha} + 2\alpha^*\frac{\partial}{\partial\alpha^*} + 2\alpha\frac{\partial}{\partial\alpha} + 4\Bigg]Q + \frac{\mathrm{i}\hbar}{m\omega}D_{pp}(t)\left[-\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \frac{1}{2}\frac{\partial^2}{\partial\alpha^{*2}} + \frac{1}{2}\frac{\partial^2}{\partial\alpha^2}\right]Q$$

$$
-\frac{\hbar}{2}(D_{xp}(t)+D_{px}(t))\left[\frac{\partial^2}{\partial\alpha^{*2}}-\frac{\partial^2}{\partial\alpha^2}\right]Q
$$

$$
-\mathrm{i}\hbar m\omega D_{xx}(t)\left[\frac{\partial^2}{\partial\alpha\partial\alpha^*}+\frac{1}{2}\frac{\partial^2}{\partial\alpha^{*2}}+\frac{1}{2}\frac{\partial^2}{\partial\alpha^2}\right]Q. \tag{61}
$$

From a comparison of equation (61) with equation (58), it is evident that the QBM is a more complicated process than the QND evolution. Writing equation (61) in polar coordinates does not simplify its structure, unlike the case of QND evolution where equation (59) was obtained in a simple form in polar coordinates. This is a reflection of the fact that QBM is a more complicated process than QND as well as the fact that in the QND case, the master equation (10) is obtained in the system energy basis which is more amenable to simplification in the Q representation (the Q-function being proportional to the diagonal element of the density matrix in the coherent state basis) than the coordinate representation in which the QBM master equation (60) is obtained.

4.2.3. Phase diffusion of the QND harmonic oscillator. We now analyze equation (59) to gain some insight into the process of phase diffusion in the case of a harmonic oscillator system coupled to its bath via a QND type of coupling. We take the long time limit. In this limit, $\dot{\eta}(t) \longrightarrow 0$ (cf remark below (22)). We solve for the Q distribution function for the zero- and high-temperature cases.

$T = 0$. In the long time limit, $\dot{\eta}(t) \longrightarrow 0$ and $\dot{\gamma}(t) \longrightarrow 0$ (cf figure 1). Then, equation (59) reduces to

$$
\frac{\partial}{\partial t}Q = \omega\frac{\partial}{\partial\theta}Q, \tag{62}
$$

which has the solution

$$
Q(\theta,t) = \mathrm{e}^{-\lambda t}\,\mathrm{e}^{-\lambda\theta/\omega}, \tag{63}
$$

where λ is a constant. Equation (62) does not have the form of a standard diffusion equation in phase space—there is a drift term but no diffusion term.

High T. In the long time limit, $\dot{\eta}(t) \longrightarrow 0$ and $\dot{\gamma}(t) \longrightarrow \gamma_0 k_B T\cosh(2r)/\hbar$ (as can be inferred from equation (31) and figure 2). Then, equation (59) becomes

$$
\frac{\partial}{\partial t}Q = \omega\frac{\partial}{\partial\theta}Q + A_1\frac{\partial^2}{\partial\theta^2}Q, \tag{64}
$$

with

$$
A_1 = \hbar\omega^2\gamma_0 k_B T\cosh(2r).
$$

This has the elementary solution

$$
Q(\theta,t) = \mathrm{e}^{-\alpha t}\,\mathrm{e}^{-A\theta}[c_1\,\mathrm{e}^{B\theta} + c_2\,\mathrm{e}^{-B\theta}], \tag{65}
$$

where α, c_1, c_2 are constants,

$$
A = \frac{\omega}{2A_1},
$$

and

$$
B = \frac{\omega}{2A_1}\sqrt{1-\frac{4\alpha A_1}{\omega^2}}. \tag{66}
$$

Equation (64) has the form of a time-dependent diffusion on a circle. It does not have the form of a pure diffusion because of the presence of an additional Kerr-like term in the master equation (56). A form similar to this arises in the phase diffusion model for the phase fluctuations of the laser field when the laser is operated far above threshold so that the amplitude fluctuations can be ignored [31]. Then the phase fluctuations due to random spontaneous emissions can be modeled as a one-dimensional random walk along the angular direction. In this sense, it can be said that the QND Hamiltonian describes diffusion of the quantum phase [40] of the light field. From equation (64) it is evident that the diffusion coefficient is dependent on the temperature T and the reservoir squeezing parameter r. In the high temperature and long time limits, the dynamical behavior is that of a quantum mechanical system influenced by an environment that is modeled by a classical stochastic process, a situation that was studied in [11]. That this is not so for the zero-T case suggests that a zero-T open quantum system cannot be simulated, even in the long time limit, by a classical stochastic bath. A detailed analysis of the phase diffusion pattern in the QND type of evolution for the two-level atomic as well as the harmonic oscillator system has been given in [41].

5. Conclusions

In this paper, we have studied the dynamics of a generic system under the influence of its environment where the coupling of the system to its environment is of the energy-preserving QND type. The bath is initially in a squeezed thermal state, decoupled from the system. We have compared the QND results with the case where the coupling is of a non-QND dissipative type for a system of two-level atom and a harmonic oscillator.

For a bosonic bath of harmonic oscillators with a QND coupling (section 2.1), we have found that in the master equation of the system, though there is a term governing decoherence, there is no dissipation term, i.e., such systems undergo decoherence without dissipation of energy. For the case where there is no squeezing in the bath, our results reduce to those obtained in [9–11] for the case of a thermal bath. The reduced density matrix of the system interacting with a bath of two-level systems (section 2.2) via a QND type of coupling is found to be independent of the temperature [9] and squeezing conditions of the bath. This brings out an intrinsic difference between a bosonic bath of harmonic oscillators and a bath of two-level systems.

We have analyzed the effect of the phase-sensitivity of the bath on the dynamics of decoherence, first by looking at the term causing decoherence in the system master equation for the bosonic bath of harmonic oscillators (obtained in section 2.1). We have evaluated the decoherence-causing term for the cases of zero and high temperatures, and also obtained its long time limit for both the cases. A study of the linear entropy $S(t)$, which is an indicator of the coherences in the reduced density matrix of the system, clearly reveals (section 3.2) that in the high-T case, the effect of the squeezing in the bath is quickly washed out and the system loses coherence over a very short time scale. In contrast, in the zero-T case, the coherences are preserved over a longer period of time and the squeezing in the bath can actually be used to improve the coherence properties of the system.

We have made a comparison between the quantum statistical mechanical processes of the QND and non-QND types of system–environment interaction. For a two-level atomic system (section 4.1), it is seen that whereas the action of the QND system–environment interaction tends to localize the system along the z-axis indicative of, in the parlance of quantum information theory, a phase-damping channel [38], the non-QND interaction (epitomized by the Lindblad equation (46)) tends to take the system towards a unique asymptotic fixed point, which for the case of zero bath-squeezing would be indicative of the (generalized) amplitude-damping channel [38]. For a harmonic oscillator system (section 4.2), we have converted the master equation to the equation for the Q representation. This brings about in a very general manner the differences in the quantum statistical mechanical processes involved in the QND and QBM. The QBM process is much more involved than the QND one in that, in addition to the decoherence and dissipation terms, it contains a number of other diffusion terms. In our analysis of the QND equation for the harmonic oscillator system, in the long time limit, we find a form similar to the one in the phase diffusion model for the fluctuations of the laser operated far above threshold when the amplitude fluctuations can be ignored. The phase fluctuations due to random spontaneous emissions can be modeled as a random walk along the angular direction. In this sense, the QND Hamiltonian describes diffusion of the quantum phase of the light field. We find that while in the high-T case the situation can be modeled as a quantum mechanical system influenced by a classical stochastic process, it is not so for the zero-T case. The high-T Q equation resembles the equation of phase diffusion on a circle which would suggest a connection between the quantum phase diffusion and QND evolution [41]. Our quantitative study provides a step towards understanding and control of the environmental impact in such open quantum systems.

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Appendix. Derivation of the reduced density matrix (equation (9))

In this appendix, we present the steps leading to the derivation of the reduced density matrix $\rho^s_{nm}(t)$ from (4) to (9). It follows from equation (8) that

$$\mathrm{e}^{\mathrm{i}H_n^{(k)}t/\hbar} = \mathrm{e}^{-\mathrm{i}E_n^2 g_k^2(\omega_k t - \sin(\omega_k t))/(\hbar^2\omega_k^2)} D\left(\frac{E_n g_k}{\hbar\omega_k}(\mathrm{e}^{\mathrm{i}\omega_k t} - 1)\right) \mathrm{e}^{\mathrm{i}\omega_k b_k^\dagger b_k t}, \tag{A.1}$$

where $D(\alpha)$ is the displacement operator,

$$D(\alpha) = \mathrm{e}^{\alpha b_k^\dagger - \alpha^* b_k}, \tag{A.2}$$

and $\sum_k H_n^{(k)} = H_n$. Similarly,

$$\mathrm{e}^{-\mathrm{i}H_n^{(k)}t/\hbar} = \mathrm{e}^{\mathrm{i}E_n^2 g_k^2(\omega_k t - \sin(\omega_k t))/(\hbar^2\omega_k^2)} D\left(\frac{E_n g_k}{\hbar\omega_k}(\mathrm{e}^{-\mathrm{i}\omega_k t} - 1)\right) \mathrm{e}^{-\mathrm{i}\omega_k b_k^\dagger b_k t}. \tag{A.3}$$

Now using equations (A.1) and (A.3) in equation (4), and making use of the following properties of the displacement operator,

$$\mathrm{e}^{\mathrm{i}\omega_k b_k^\dagger b_k t} D(\alpha) = D(\alpha\,\mathrm{e}^{\mathrm{i}\omega_k t})\,\mathrm{e}^{\mathrm{i}\omega_k b_k^\dagger b_k t}, \tag{A.4}$$

$$D^\dagger(\alpha) = D(-\alpha), \tag{A.5}$$

$$D^\dagger(\alpha) D(\alpha\,\mathrm{e}^{\mathrm{i}\omega_k t}) = D(\alpha(\mathrm{e}^{\mathrm{i}\omega_k t} - 1))\,\mathrm{e}^{\mathrm{i}\alpha\alpha^*\sin(\omega_k t)}, \tag{A.6}$$

the reduced density matrix in the system eigenbasis becomes

$$\rho^s_{nm}(t) = \mathrm{e}^{-\mathrm{i}(E_n - E_m)t/\hbar}\mathrm{e}^{-\mathrm{i}(E_n^2 - E_m^2)\sum_k (g_k^2 \sin(\omega_k t)/\hbar^2\omega_k^2)} \prod_k \mathrm{Tr_R}[\rho_R(0) D(\theta_k)]\rho^s_{nm}(0). \tag{A.7}$$

Here,

$$\theta_k = (E_m - E_n)\frac{g_k}{\hbar\omega_k}(\mathrm{e}^{\mathrm{i}\omega_k t} - 1), \tag{A.8}$$

and $\rho_R(0)$ is as in equation (5).

The trace term in equation (A.7) is

$$\begin{aligned}\prod_k \mathrm{Tr_R}[\rho_R(0)D(\theta_k)] &= \prod_k \mathrm{Tr_R}[S(r_k,\Phi_k)\rho_{\mathrm{th}}S^\dagger(r_k,\Phi_k)D(\theta_k)]\\ &= \prod_k \mathrm{Tr_R}[\rho_{\mathrm{th}}D(\theta_k\cosh(r_k) + \theta_k^*\sinh(r_k)\,\mathrm{e}^{2\mathrm{i}\Phi_k})]\\ &= \exp\left[-\frac{1}{2}(E_m - E_n)^2\sum_k \frac{g_k^2}{\hbar^2\omega_k^2}\coth\left(\frac{\beta\hbar\omega_k}{2}\right)\right.\\ &\quad\left.\times\,|(\mathrm{e}^{\mathrm{i}\omega_k t} - 1)\cosh(r_k) + (\mathrm{e}^{-\mathrm{i}\omega_k t} - 1)\sinh(r_k)\,\mathrm{e}^{2\mathrm{i}\Phi_k}|^2\right].\end{aligned} \tag{A.9}$$

Here we have used the following relation between the squeezing and displacement operators:

$$S^\dagger(r_k,\Phi_k)D(\theta_k)S(r_k,\Phi_k) = D(\theta_k\cosh(r_k) + \theta_k^*\sinh(r_k)\,\mathrm{e}^{2\mathrm{i}\Phi_k}). \tag{A.10}$$

Using equation (A.9) in equation (A.7), the reduced density matrix $\rho^s_{nm}(t)$ (9) is obtained.

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